Exact WKB analysis for Schrodinger equations with periodic potentials

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Tatsuya Koike

1 Introduction.

Let us consider one-dimensional Schrödinger equations,

$$\left(-\frac{d^2}{dz^2}+\eta^2 Q(\cos z)\right)\phi=0, \tag{1}$$

and

$$\left(-\frac{d^2}{dx^2}+\eta^2\left(\frac{Q(x)}{1-x^2}-\eta^{-2}\frac{x^2+2}{4(1-x^2)^2}\right)\right)\psi=0, \tag{2}$$

where η is a large parameter, and $Q(x)$ is a polynomial in x. These equations are related via the transformation,

$$\begin{aligned} x &= \cos z, & (3)\\ \psi &= \sqrt{\sin z}\phi. & (4)\end{aligned}$$

Because this transformation (3) is a singular one, that is, its first derivative vanishes at $z=k\pi$, there appear singularities in (2) at $x=\pm 1$.

Our interest is to analyze these equations from the viewpoint of exact WKB analysis, in which the Stokes geometry (turning points, Stokes curves, singularities, ...) plays an important role to grasp the global behavior of solutions, and the interesting point in the above example is the Stokes geometry of (1) could not correspond to that of (2) due to the singularities $x=\pm 1$ of (2). To make a further reasoning, we need to clarify the role of $x=\pm 1$ in WKB analysis. (We already know the resurgent structure of (1) with respect to η due to the work given by E.Delabaere [D].)

In this note, motivated by this example, we will mainly be concerned with

$$\left(-\frac{d^2}{d\widetilde{x}^2}+\eta^2\left(\frac{A(\widetilde{x})}{\widetilde{x}}+\eta^{-2}\frac{B(\widetilde{x})}{\widetilde{x}^2}\right)\right)\widetilde{\psi}=0, \tag{5}$$

near $\widetilde{x}=0$, where $A(\widetilde{x})$ and $B(\widetilde{x})$ are holomorphic functions at $\widetilde{x}=0$, and we also assume $A(0)\neq 0$. Our purpose here is

(i) to confirm that we must consider $\widetilde{x}=0$ as a turning point (Hence one Stokes curve emanates from $\widetilde{x}=0$.);

(ii) to determine the connection formula at $\widetilde{x}=0$,

from a viewpoint of exact WKB analysis.

Thus $x=\pm 1$ are not only the singular points but also the turning points of (2), while the corresponding points $z=k\pi$ $(k\in\mathbb{Z})$ are not. Hence the correspondence of the Stokes geometry between (1) and (2) does not hold at all, which leads us to the expectation that the turning points $x=\pm 1$ may be apparent. We can prove this expectation is true as a corollary of our result; A turning point could become an apparent one if this point is a singular point of the *subleading term* of the potential. This is the second point we wish to emphasize in this note. We will explain its details in the last section, together with some examples of Stokes curves for (1) and (2).

In what follows we will use the same notations and definitions as in [AKT1], such as WKB solutions, their Borel sum, etc. See [AKT1] or [DP] (and, of course, [V]) and the references cited there for the explanation of exact WKB analysis.

2 A simple pole as a turning point.

2.1 The canonical equation.

Let us start with the following simple, but important equation, which is called a canonical equation of (5) at $\widetilde{x}=0$ (just like Airy equation near a simple turning point):

$$\left(-\frac{d^2}{dx^2}+\eta^2\frac{1}{x}+\frac{b}{x^2}\right)\psi=0, \tag{6}$$

where b is some constant, independent of η. WKB solutions of this equation are

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_0^x S_{\mathrm{odd}} dx\right). \tag{7}$$

where $S_{\mathrm{odd}} = \eta S_{-1} + \eta^{-1} S_1 + \eta^{-3} S_3 + \cdots$, is the odd degree part of the solution $S = \eta S_{-1} + S_0 + \eta^{-1} S_1 + \cdots$, of the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 \frac{1}{x} + \frac{b}{x^2}. \tag{8}$$

Explicitly we obtain

$$S = \eta \frac{1}{\sqrt{x}} + \frac{1}{4x} + \eta^{-1} \frac{16b+3}{32\sqrt{x^3}} + \eta^{-2} \frac{16b+3}{64x^2} + \cdots. \tag{9}$$

Note that we can show, by induction, that

$$S_j = (const.)\, x^{-\frac{1}{2}j-1} \tag{10}$$

holds, namely, the degree of singularities of S_j at $x = 0$ becomes higher and higher as j tends to infinity, which is the common property with "usual" turning points. This is the intuitive reason why we call $x = 0$ as a turning point.

In the following we will fix the branch of $\sqrt{x}$ such that

$$\sqrt{x} > 0 \ \text{ for } \ x > 0 \tag{11}$$

holds.

Now we show

Proposition 1 *WKB solutions $\psi_{\pm}$ of (6) are Borel summable except the positive real axis. Let $\widehat{\psi}_{\pm}$ be analytic continuation of $\psi_{\pm}$ across the positive real axis from the lower half plane to the upper. Then we obtain*

$$\widehat{\psi}_{+} = \psi_{+} + 2i \cos \pi\sqrt{1+4b}\, \psi_{-}, \tag{12}$$
$$\widehat{\psi}_{-} = \psi_{-}. \tag{13}$$

We note here that $\sqrt{1+4b}$ is the difference of two characteristic exponents of (6) at $x = 0$.

Proof. The proof is a standard one (see [AKT2], [V]), but we shall follow the proof in detail because of its importance to understand the meaning of a turning point and a Stokes curve.

Because of (10), we can expand the WKB solutions $\psi_\pm$ as

$$\psi_\pm = \sqrt{x}\exp\left(\pm\eta\int_0^x S_{-1}dx\right)\sum_{j=0}^{\infty}\psi_{\pm,j}\left(\eta\sqrt{x}\right)^{-j-1/2}, \tag{14}$$

where $\psi_{\pm,j}$ are constants independent of η and x. Hence their Borel transform are of the following form:

$$\psi_{\pm,B}(x,y) = \sum_{j=0}^{\infty}\frac{\psi_{\pm,j}}{\Gamma(j+1/2)}\left(\frac{y}{\sqrt{x}}\pm 2\right)^{j-1/2}, \tag{15}$$

that is, $\psi_{\pm,B}(x,y)$ depends on x and y through

$$s = \frac{1}{4}\left(\frac{y}{\sqrt{x}}+2\right),\quad 1-s = \frac{1}{4}\left(-\frac{y}{\sqrt{x}}+2\right), \tag{16}$$

respectively. In addition, $\psi_{\pm,B}(x,y)$ satisfy

$$\left(-\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial^2}{\partial y^2}+\frac{b}{x^2}\right)\psi_{\pm,B}(x,y) = 0. \tag{17}$$

Hence we find that

$$\left(s(1-s)\frac{d^2}{dx^2}+\left(\frac{3}{2}-3s\right)\frac{d}{ds}+4b\right)\psi_{\pm,B} = 0 \tag{18}$$

holds, which enables us to determine $\psi_{\pm,B}$ explicitly as follows:

$$\psi_{+,B}(x,y) = \frac{1}{\sqrt{4\pi}}s^{-1/2}\,F\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\frac{1}{2};s\right)\Bigg|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}}, \tag{19}$$

$$\psi_{-,B}(x,y) = \frac{1}{\sqrt{-4\pi}}(1-s)^{-1/2}\,F\left(\frac{3}{2}-\alpha,\frac{3}{2}-\beta,\frac{1}{2};1-s\right)\Bigg|_{s=\frac{y}{4\sqrt{x}}+\frac{1}{2}}. \tag{20}$$

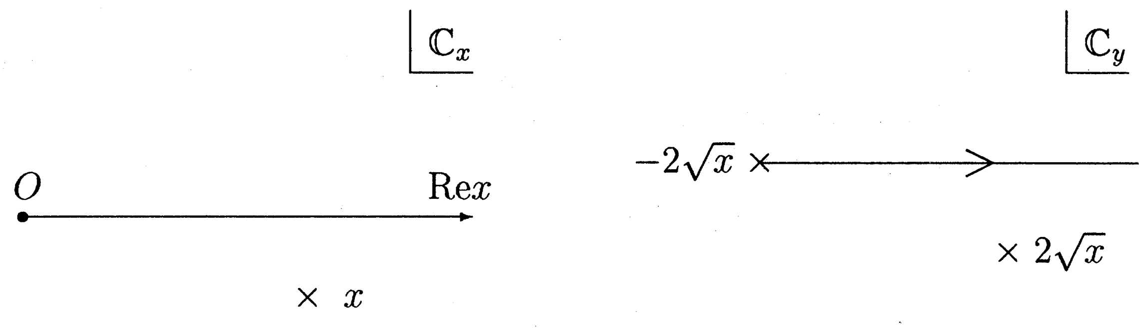

Figure 1: If x is below the positive real axis in x-plane (left), the location of singularities (indicated by $\times$) are as shown in the right figure. The straight line in the right figure indicates the contour of Borel sum.

where F denotes the Gauss hypergeometric function and α, β are constants satisfying

$$\alpha + \beta = 2, \tag{21}$$
$$\alpha\beta = -4b. \tag{22}$$

Because the singularities of $\psi_{\pm,B}(x,y)$ are

$$s = 0,\ 1 \quad \Leftrightarrow \quad y = -2\sqrt{x},\ 2\sqrt{x}, \tag{23}$$

its Borel sum

$$\psi_{\pm}(x,\eta) = \int_{\mp 2\sqrt{x}}^{\infty} e^{-\eta y}\psi_{\pm,B}(x,y)dy, \tag{24}$$

where the integral are performed in parallel with the positive real axis, are well-defined except

$$\mathrm{Im}\left(2\sqrt{x}\right) = \mathrm{Im}\left(-2\sqrt{x}\right), \tag{25}$$

namely, except the positive real axis, which is called a Stokes curve (or line) emanating from a turning point $x = 0$. (See Fig. 1.)

Let x be below the positive real axis as shown in the left of Fig.1. The right of Fig.1 indicates the contour of the Borel sum of ψ_+ and the singularities of $\psi_{+,B}(x,y)$ in the y-plane.

If x moves to the upper half plane across the positive real axis as shown in the left of Fig.2, we must deform the contour of Borel sum to C as shown in the right of Fig.2, because the singularities $y = \pm 2\sqrt{x}$ move as x moves. Hence we obtain

$$\widehat{\psi}_+(x,\eta) = \int_C e^{-\eta y}\psi_{+,B}(x,y)dy. \tag{26}$$

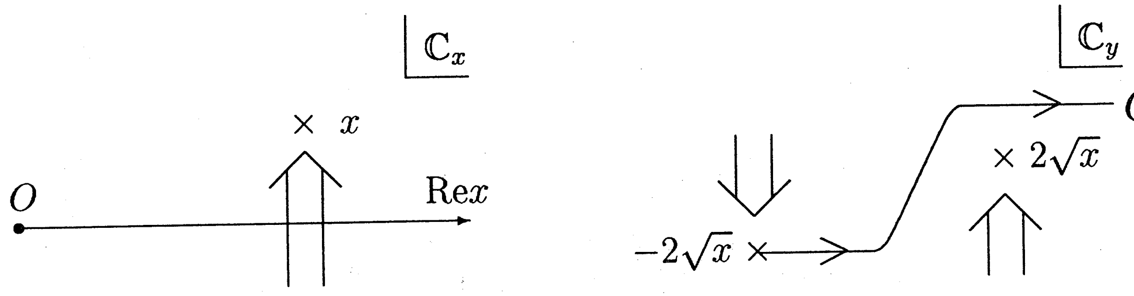

Figure 2: If x moves as in the left, the singularities also move. We must deform the contour to C.

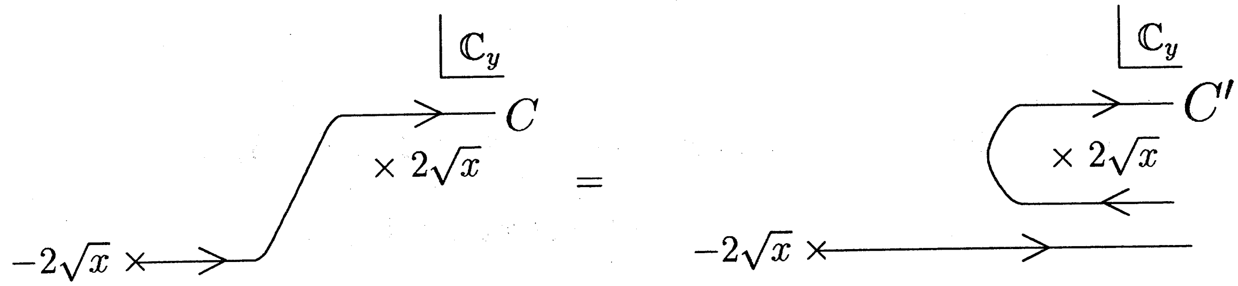

Figure 3: The integration along C is equal to sum of the integration along the contours indicated in the right figure.

This is equal to (See Fig.3.)

$$\int_{-2\sqrt{x}}^{\infty} e^{-\eta y}\psi_{+,B}(x,y)dy + \int_{C'} e^{-\eta y}\psi_{+,B}(x,y)dy. \tag{27}$$

By making use of the connection formula of hypergeometric functions

$$\begin{aligned} & s^{-1/2}F\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\frac{1}{2};s\right) \\ = & \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(1-\alpha)\Gamma(1-\beta)}F\left(\alpha,\beta,\frac{3}{2};1-s\right) \\ & +\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\right)}(1-s)^{-1/2}F\left(\frac{3}{2}-\alpha,\frac{3}{2}-\beta,\frac{1}{2};1-s\right), \end{aligned} \tag{28}$$

we obtain

$$
\begin{aligned}
& \int_{C'} e^{-\eta y}\psi_{+,B}(x,y) \\
= \ & 2i\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\right)}\int_{2\sqrt{x}}^{\infty} e^{-\eta y}\psi_{-,B}(x,y)dy && (29) \\
= \ & 2i\cos\pi\sqrt{1+4b}\,\psi_{-}(x,\eta). && (30)
\end{aligned}
$$

This prove the proposition. □

In the following, for simplicity, we denote (12) and (13) as

$$
\begin{aligned}
\psi_{+} &\mapsto \psi_{+} + 2i\cos\pi\sqrt{1+4b}\,\psi_{-}, && (31) \\
\psi_{-} &\mapsto \psi_{-}. && (32)
\end{aligned}
$$

2.2 Connection formulae at a simple pole.

Let $\widetilde{\psi}_{\pm}$ be WKB solutions of (5):

$$\widetilde{\psi}_{\pm} = \frac{1}{\sqrt{\widetilde{S}_{\mathrm{odd}}}}\exp\left(\pm\int_0^{\widetilde{x}}\widetilde{S}_{\mathrm{odd}}d\widetilde{x}\right), \qquad (33)$$

where

$$\widetilde{S} = \eta\sqrt{\frac{A(\widetilde{x})}{\widetilde{x}}} + \frac{1}{4}\left(\frac{1}{\widetilde{x}} - \frac{A'(\widetilde{x})}{A(\widetilde{x})}\right) + \cdots. \qquad (34)$$

Here, for brevity, we use the following notation:

$$\int_0^{\widetilde{x}} = \frac{1}{2}\int_{C_{\widetilde{x}}}, \qquad (35)$$

where $C_{\widetilde{x}}$ is the contour in $\widetilde{x}$-plane as shown in Fig.4. (Note that

$$S_{\mathrm{odd},j}(\widetilde{x}) = R_j(\widetilde{x})\,\widetilde{x}^{-\frac{1}{2}j-1} \qquad (36)$$

holds, where $R_j(\widetilde{x})$ is a holomorphic function at $\widetilde{x}=0$.)

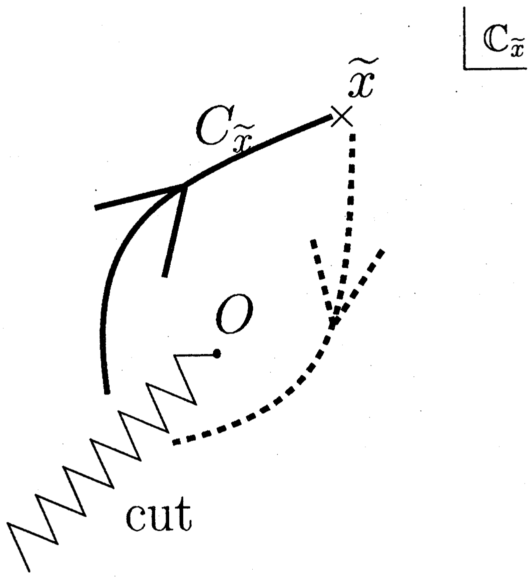

Figure 4: The contour C_x. We place a cut (indicated by the wiggly line) for S_{odd}.

Let Γ be a (real one-dimensional) curve defined by

$$\mathrm{Im} \int_0^{\widetilde{x}} \sqrt{\frac{A(\widetilde{x})}{\widetilde{x}}} d\widetilde{x} = 0. \tag{37}$$

We call this curve as a Stokes curve emanating from $\widetilde{x} = 0$. In this subsection we show the following:

Theorem 2 *Borel sum of WKB solutions $\widetilde{\psi}_\pm$ of (5) are well-defined near $\widetilde{x} = 0$ except Γ. Crossing this curve Γ in a counterclockwise manner with respect to the center $\widetilde{x} = 0$, we obtain*

$$\widetilde{\psi}_+ \mapsto \widetilde{\psi}_+ + 2i \cos \pi \sqrt{1 + 4B(0)}\, \widetilde{\psi}_-, \tag{38}$$

$$\widetilde{\psi}_- \mapsto \widetilde{\psi}_-, \tag{39}$$

when

$$\mathrm{Re} \int_0^{\widetilde{x}} \sqrt{\frac{A(\widetilde{x})}{\widetilde{x}}} d\widetilde{x} > 0. \tag{40}$$

(In this case we call $\widetilde{\psi}_+$ is dominant on Γ.) In the case

$$\mathrm{Re} \int_0^{\widetilde{x}} \sqrt{\frac{A(\widetilde{x})}{\widetilde{x}}} d\widetilde{x} < 0, \tag{41}$$

we replace $\psi_\pm$ to $\psi_\mp$ in (38) and (39) respectively.

We note here that

(i) the Stokes multiplier $2i\cos\pi\sqrt{1+4B(0)}$ is independent of $A(\widetilde{x})$. Roughly speaking, the leading term $A(\widetilde{x})/x$ of the potential determines the shape of Stokes geometry, while the lower order term $B(\widetilde{x})/\widetilde{x}^2$ determines its Stokes multiplier;

(ii) $\sqrt{1+4B(0)}$ is the difference of two characteristic exponents at $\widetilde{x}=0$.

We give the proof of Theorem 2 based on *the transformation theory* developed in [AKT2]. The first step in the proof is to show the following proposition:

Proposition 3 *There exists an infinite series*

$$\begin{aligned} x &= x(\widetilde{x},\eta) \\ &= x_0(\widetilde{x})+\eta^{-1}x_1(\widetilde{x})+\cdots, \end{aligned} \tag{42}$$

such that

(i) each $x_j(\widetilde{x})$ are holomorphic at $\widetilde{x}=0$, satisfying $x_0(0)=0$ and $x_0'(0)\neq 0$;

(ii) The following relation holds:

$$\frac{A(\widetilde{x})}{\widetilde{x}}+\eta^{-2}\frac{B(\widetilde{x})}{\widetilde{x}^2}=\left(\frac{\partial x}{\partial\widetilde{x}}\right)^2\left(\frac{1}{x}+\eta^{-2}\frac{B(0)}{x^2}\right)-\frac{1}{2}\eta^{-2}\{x;\widetilde{x}\}. \tag{43}$$

Here $'$ denotes the differentiation with respect to $\widetilde{x}$, and

$$\{x;\widetilde{x}\}=\frac{x'''}{x'}-\frac{3}{2}\left(\frac{x''}{x'}\right)^2, \tag{44}$$

the Schwartzian derivative.

Moreover we can show that $x=x(\widetilde{x},\eta)$ is pre-Borel-summable and $x_j(0)=0$ holds for $j=1,2,\cdots$.

This proposition enables us to reduce (5) to (6), e.g., we find

$$\widetilde{\psi}_\pm(\widetilde{x},\eta)=\left(\frac{\partial x}{\partial\widetilde{x}}\right)^{-1/2}\psi_\pm(x(\widetilde{x},\eta),\eta). \tag{45}$$

Proof of the proposition. First, we construct two infinite series (42) and

$$b = b_0 + \eta^{-1} b_1 + \cdots, \tag{46}$$

where b_j are some constants, satisfying the condition (i) and

$$\frac{A(\tilde{x})}{\tilde{x}} + \eta^{-2}\frac{B(\tilde{x})}{\tilde{x}^2} = \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 \left(\frac{1}{x} + \eta^{-2}\frac{b}{x^2}\right) - \frac{1}{2}\eta^{-2}\{x;\tilde{x}\}. \tag{47}$$

By substituting (42) into (47), and comparing both sides degree by degree, we obtain

$$\frac{A(\tilde{x})}{\tilde{x}} = \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 \frac{1}{x_0} \tag{48}$$

for the 0-th degree, and

$$\left(\frac{2x_0'}{x_0}\frac{d}{d\tilde{x}} - \left(\frac{x_0'}{x_0}\right)^2\right) x_j(\tilde{x}) = F_j(\tilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 b_{j-2}, \tag{49.j}$$

for the j (≥ 1)-th degree, where $b_{-1} = 0$ and

$$\begin{aligned} F_1(x) &= 0, & (50)\\ F_2(x) &= \frac{B(\tilde{x})}{\tilde{x}} - \frac{(x_0' x_1)^2}{x_0{}^3} + 2\frac{x_0' x_1' x_1}{x_0{}^2} + \frac{1}{2}\{x_0;\tilde{x}\}, & (51) \end{aligned}$$

and

$$\begin{aligned} F_n(x) &= \sum_{\substack{k_1+k_2+\mu+l=n\\ \mu_1+\cdots+\mu_l=\mu}} (-1)^{l+1} x_{k_1}' x_{k_2}' \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{x_0{}^{l+1}} \\ &\quad + \sum_{\substack{k_1+k_2+m+\mu=n-2\\ \mu_1+\cdots\mu_l=\mu}} (-1)^{l+1}(l+1) b_m x_{k_1}' x_{k_2}' \frac{x_{\mu_1+1}\cdots x_{\mu_l+1}}{x_0{}^{l+2}} \\ &\quad + \frac{1}{2}\{x;\tilde{x}\}_{l+2}. \end{aligned} \tag{52.n}$$

Here we denote by $\{x;\tilde{x}\}_n$ the n-th degree part of $\{x;\tilde{x}\}$:

$$\{x;\tilde{x}\} = \sum_{n=0}^{\infty} \eta^{-n}\{x;\tilde{x}\}_n. \tag{53}$$

The holomorphic solutions of (48) and (49.1) are

$$\begin{aligned} x_0(\widetilde{x}) &= \frac{1}{4}\left(\int_0^{\widetilde{x}} \sqrt{\frac{A(\widetilde{x})}{\widetilde{x}}} d\widetilde{x}\right)^2 && (54) \\ &= A(0)x + \cdots, && (55) \\ x_1(\widetilde{x}) &= 0, && (56) \end{aligned}$$

which satisfy the condition (i) and (ii).

The holomorphic solution of (49.2) is

$$x_2(\widetilde{x}) = \sqrt{x_0} \int_0^{\widetilde{x}} \frac{\sqrt{x_0}}{2x_0'} \left(\frac{B(\widetilde{x})}{\widetilde{x}^2} - \left(\frac{x_0'}{x_0}\right)^2 b_0 + \frac{1}{2}\{x_0; \widetilde{x}\} \right) d\widetilde{x}. \qquad (57)$$

We chose $b_0 = B(0)$ to ensure $x_2(0) = 0$; Otherwise $x_2(\widetilde{x})$ does not vanish at $\widetilde{x} = 0$, and $F_4(\widetilde{x})$ (or, at least, $F_6(\widetilde{x})$) has a pole of degree ≥ 3. In this case (49.4) (or (49.6)) does note have a solution which satisfies (i).

We can solve (49.j) ($j \geq 3$) recursively in a similar fashion:

$$x_j(\widetilde{x}) = \sqrt{x_0} \int_0^{\widetilde{x}} \frac{\sqrt{x_0}}{2x_0'} \left(F_j(\widetilde{x}) - \left(\frac{x_0'}{x_0}\right)^2 b_{j-2} \right) d\widetilde{x}, \qquad (58.j)$$

and

$$b_{j-2} = \widetilde{x}^2 F_j(\widetilde{x})\big|_{\widetilde{x}=0}. \qquad (59.j)$$

Note that $F_j(x)$ has a (at most) double pole at the origin because $x_0(\widetilde{x}), \cdots, x_{j-1}(\widetilde{x})$ vanish at $\widetilde{x} = 0$. We choose (59.j) to ensure $x_j(\widetilde{x})$ vanish at $\widetilde{x} = 0$; Otherwise $F_{j+2}(\widetilde{x})$ has a pole of degree ≥ 3 and (49) does note have a solution which satisfies (i).

We have thus constructed $x(\widetilde{x}, \eta)$ and b as desired.

Next, by multiplying both side of (47) by $\widetilde{x}^2$, and taking the limit $\widetilde{x}$ tends to 0, we obtain

$$\begin{aligned} \eta^{-2} B(0) &= \lim_{\widetilde{x} \to 0} \left(\frac{\partial x}{\partial \widetilde{x}}\right)^2 \left(\frac{\widetilde{x}}{x}\right)^2 \eta^{-2} b && (60) \\ &= \eta^{-2} b, && (61) \end{aligned}$$

because $x_j(0) = 0$. Hence we obtain

$$b_0 = B(0), b_j = 0 \ \text{ for } j \geq 1. \qquad (62)$$

What remains is to show the pre-Borel-summability of (42) we have constructed so far, and we can show that there exist constants $r, C > 0$ such that

$$\sup_{|\widetilde{x}|\le r-\epsilon} |x_n(\widetilde{x})|, \ \sup_{|\widetilde{x}|\le r-\epsilon} |x_n'(\widetilde{x})|, \ \sup_{|\widetilde{x}|\le r-\epsilon} \left|\frac{x_n(\widetilde{x})}{x_0(\widetilde{x})}\right| \le n!\, C^{n-1}\, \epsilon^{-n}, \tag{63.n}$$

for any ϵ satisfying $0 < \epsilon < r$, but we omit the details here. (Cf. [AKT2].)
□

We give one remark that we can show, by induction, that

$$x_{2j+1}(\widetilde{x}) \equiv 0, \tag{64}$$

holds for $j = 0, 1, 2, \cdots$.

By expanding the right hand side of (45), we obtain

$$\widetilde{\psi}_{\pm}(\widetilde{x}, \eta) = \left(\frac{\partial x}{\partial \widetilde{x}}\right)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=1}^{\infty} x_j \eta^{-j}\right)^n \left(\frac{\partial}{\partial x}\right)^n \psi_{\pm}(x, \eta)\Bigg|_{x=x_0(\widetilde{x})} . \tag{65}$$

Hence, after the Borel transform, we obtain,

$$\begin{aligned} \widetilde{\psi}_{\pm,B}(\widetilde{x}, y) &= \left(\frac{\partial x(\widetilde{x}, \partial_y)}{\partial \widetilde{x}}\right)^{-1/2} \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=1}^{\infty} x_j \left(\frac{\partial}{\partial y}\right)^{-j}\right)^n \left(\frac{\partial}{\partial x}\right)^n \psi_{\pm,B}(x, y)\Bigg|_{x=x_0(\widetilde{x})} \end{aligned} \tag{66}$$

By using this relation (together with the pre-Bore-summability of $x = x(\widetilde{x}, \eta)$) and Proposition 1, we can prove Theorem 2. (We omit the details here.)

3 Application to the periodic potential.

Let us now return to analyze (1) and (2) by making use of the result obtained so far. In the following we will assume that $Q(\pm 1) \neq 0$ holds to ensure that $x = \pm 1$ are simple poles of the leading term of the potential.

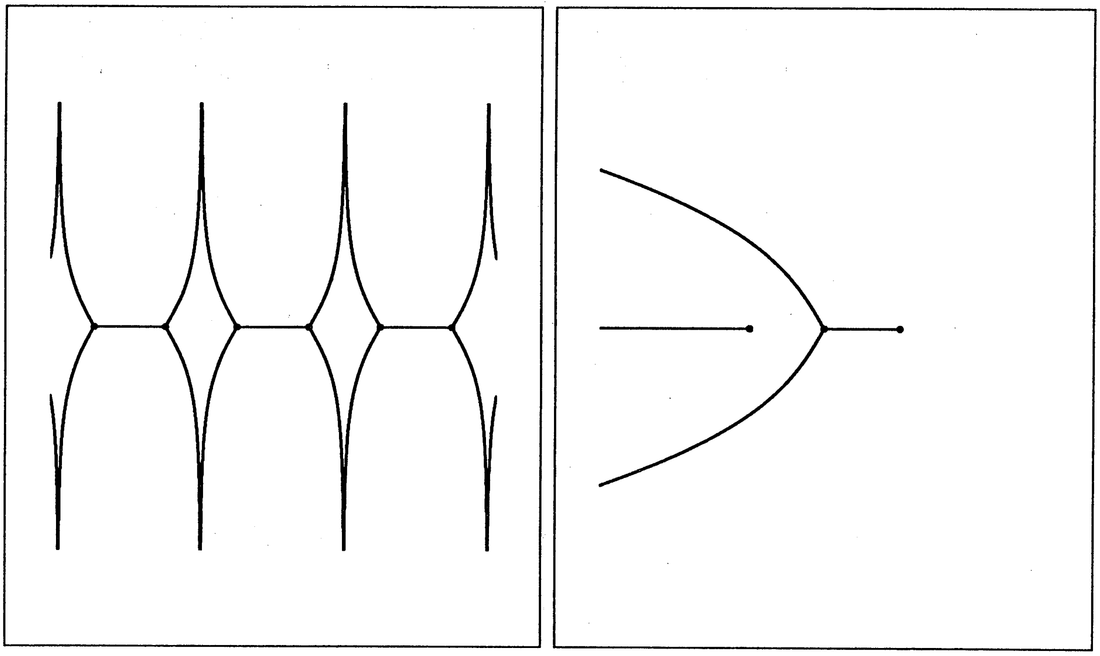

Figure 5: The Stokes curves for (1) (left) and for (2) (right) when $Q(x) = x$.These Stokes curves are degenerate.

First of all we will see some examples of the Stokes geometry of (1) and (2).

As the first example, we take $Q(x) = x$. In this case turning points of (1) are

$$z = (k + \frac{1}{2})\pi \ : \text{simple turning points} \tag{67}$$

where $k \in \mathbb{Z}$. See the left of Fig.5 for the Stokes curves of (1), which are defined by

$$\operatorname{Im} \int_a^z \sqrt{\cos z}\,dz = 0, \tag{68}$$

where a is a turning point.

On the other hand, turning points of (2) are

$$x = \begin{cases} 0 & : \text{ a simple turning point,} \\ \pm 1 & : \text{ simple poles.} \end{cases} \tag{69}$$

See the right of Fig.5 for Stokes curves of (2), which are defined by

$$\operatorname{Im} \int_a^x \sqrt{\frac{x}{1-x^2}}\,dx = 0, \tag{70}$$

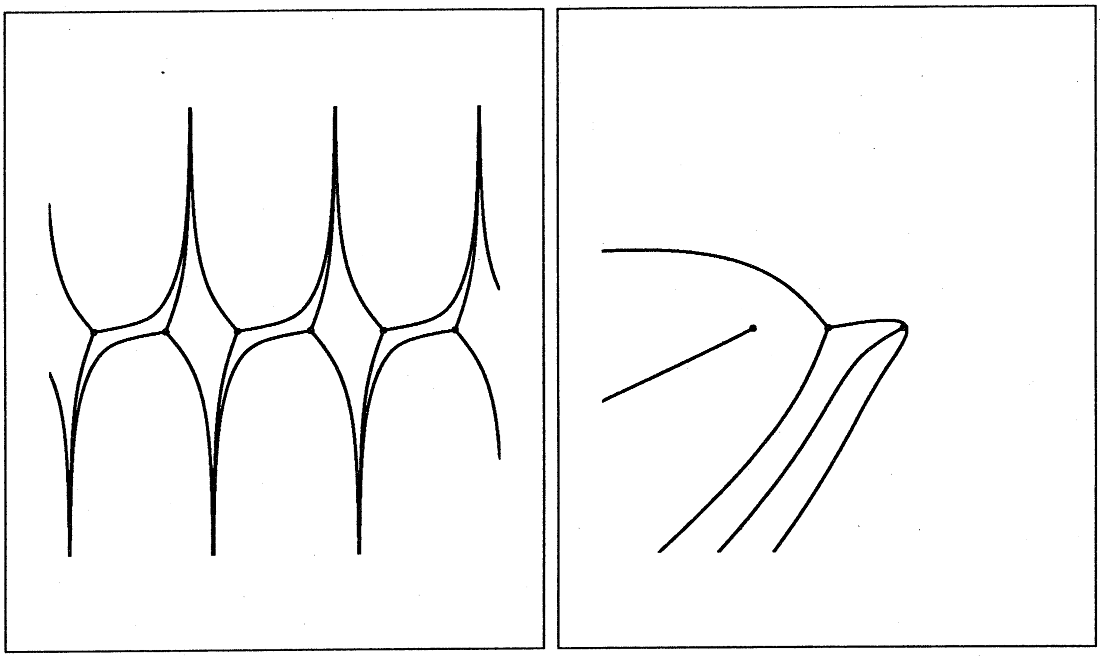

Figure 6: We resolve the degeneracy in Fig.4 by changing η to $e^{i\theta}\,\eta$ for $\theta = -1/2$.

where a is a turning point.

These Fig.5 are, however, degenerate, namely, there are Stokes curves connecting two turning points. (In this case, there appear singularities on the contour of the Borel sum. See [DP].) We resolve this degeneracy by changing η to $e^{i\theta}\eta$, where $\theta < 0$ is sufficiently small. Then Stokes curves are as in Fig.6.

By the transformation $x = \cos z$, simple turning points of (1) are mapped to a simple turning point (2), while the normal points $z = k\pi$ $(k \in \mathbb{Z})$ of (1) are mapped to $x = \pm 1$. Hence the correspondence of the Stokes geometry does note hold, as we observed in Introduction.

As the second example, we take $Q(x) = x + t$, where $t > 1$. In this case, turning points of (1) are

$$z = \pm\tau + (2k+1)\pi \ : \ \text{simple turning points} \tag{71}$$

where $t = \cos\tau$ and $k \in \mathbb{Z}$. See the left of Fig.7 for the Stokes curves. Turning points of (2) are

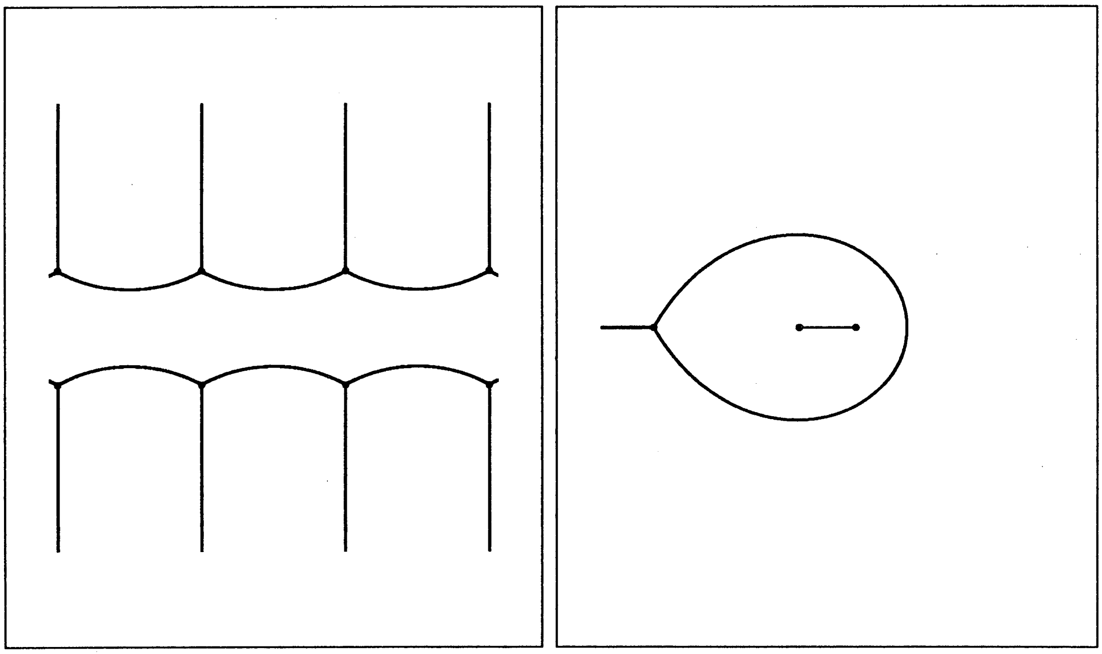

Figure 7: The Stokes curves for (1) (left) and for (2) (right) when $Q(x) = x + 2$.These Stokes curves are degenerate.

$$x = \begin{cases} -t & : \text{ a simple turning pints,} \\ \pm 1 & : \text{ simples poles,} \end{cases} \tag{72}$$

and their Stokes curves are as in the right of Fig.7. These Stokes curves are also degenerate. We try to resolve this degeneracy in the same manner as in the above example. For generic $\theta < 0$, we find that we can resolve this degeneracy of the Stokes geometry. But, what is interesting is that the Stokes geometry becomes degenerate infinitely many times, as θ tends to 0. (See Fig.8, 9.)

In any example, the correspondence of the Stokes geometry does note hold between (1) and (2).

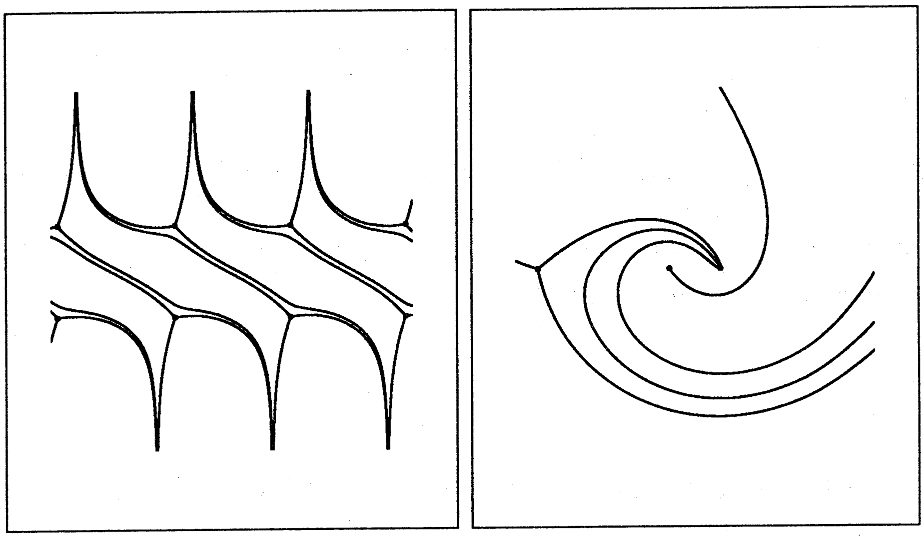

Figure 8: We resolve the degeneracy in Fig.6 by changing η to $e^{i\theta}\,\eta$ for $\theta = -1$.

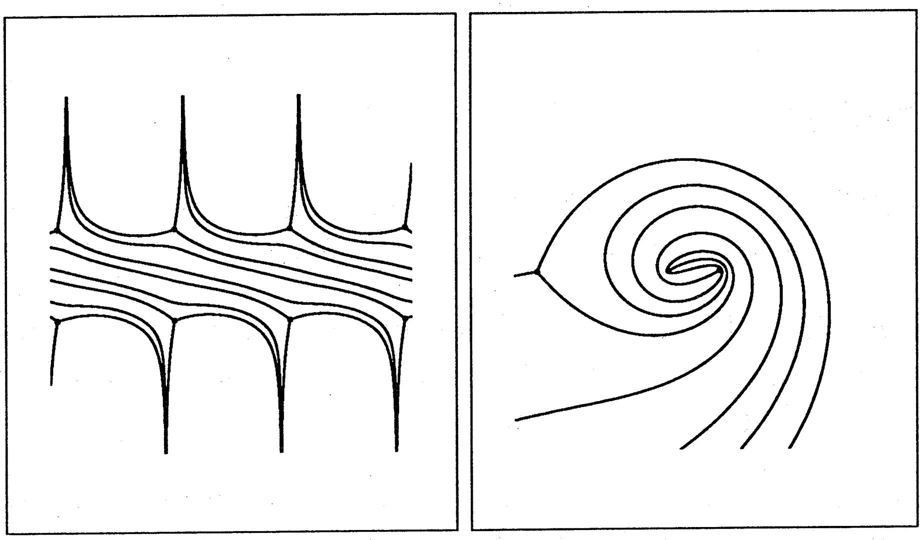

Figure 9: We resolve the degeneracy in Fig.6 by changing η to $e^{i\theta}\,\eta$ for $\theta = -1/2$. By comparing this figure with Fig.7, we find that the Stokes geometry becomes degenerate between $-1 < \theta < -1/2$. Furthermore, we find that the Stokes geometry becomes degenerate infinitely many times as θ tends to 0.

Now we apply Theorem 2 to (2) at $x = \pm 1$. We must note that

(i) the connection formula (2) depends only on the lower degree part of the potential;

(ii) The lower degree part of the potential are independent of $Q(x)$.

Thus the connection formulae at $x = \pm 1$ do not depend on $Q(x)$.

Characteristic exponents of (2) at $x = \pm 1$ are

$$\frac{3}{4}, \frac{1}{4}. \tag{73}$$

Hence the Stokes multiplier which arises when we cross the Stokes curves emanating from $x = \pm 1$ vanishes because

$$2i \cos \pi \left(\frac{3}{4} - \frac{1}{4} \right) = 0 \tag{74}$$

holds, which implies the turning points $x = \pm 1$ are apparent, as we expected.

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